

Charged States In A Semiclassical Model Of Infrared Gupta-Bleuler Quantum Electrodynamics

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We address the problem of the identification and characterization of charged states within local and covariant quantizations of abelian gauge theories, focusing on a semiclassical model of infrared Gupta-Bleuler Quantum Electrodynamics, based on the Bloch-Nordsieck approximation and formulated in Feynman's gauge. The *GNS* construction over suitable functionals yields positive subspaces of the indefinite-metric space of the model; charged states with Liénard-Wiechert space-like asymptotics can then be constructed via an automorphism of the algebra of observables implementing Gauss' law. Finally, by an analysis of the localization properties of the corresponding expectations over the asymptotic electromagnetic fields, it is shown that such states identify an infrared-minimal charge class in the sense of Buchholz.

Keywords: infrared problem, quantum electrodynamics, local gauge quantization, solvable models, solvable models, charged states, Gauss' law

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INTRODUCTION

One of the main open questions in the theoretical understanding of Quantum Electrodynamics (QED) concerns the construction and the characterization of physical charged states. This issue is much more difficult, with respect to a standard Quantum Field Theory (QFT), owing to the presence of a local Gauss law. The latter relates the total charge in a bounded region to the flux of electric field through the boundary of the region and by locality of observables can be shown to imply, within a quantum mechanical setting, the non locality of the electrically charged states¹ and the superselection of electric charge². By the same token, it implies that the state space of QED contains uncountably many superselection sectors, labeled by the value of the electric flux in suitably chosen spacelike cones^{3,4}, non-invariant under Lorentz boosts⁴⁻⁶ and with single-particle subspaces not containing proper eigenstates of the mass operator⁴.

Besides being a fundamental issue in the understanding of the structural properties of (abelian) gauge theories, the identification of physical charged states is also necessary for the solution of the infrared problem, namely for the characterization of states of charged particles and of the quantized e.m. field at asymptotic times and for the definition of a scattering matrix. In particular, the question arises whether it is possible to formulate a collision theory in terms of an appropriate subset of charged states; in this respect, it is plausible that the discussion of the infrared problem may be made simpler by choosing the (charged) sectors with the best localization properties relative to the vacuum.

Quite generally, whatever formulation one may employ, the construction of the charged sectors of QED always presents subtle features. Within the Haag-Kastler formulation⁷, based on nets of algebras of local observables, charged sectors should in principle be obtained acting on the vacuum sector with suitable morphisms of the observable algebra, by a generalization of the construction of Doplicher, Haag and Roberts⁸; however, since the needed morphism cannot be local, by virtue of Gauss' constraint, such a generalization is difficult to be realized and has not yet been accomplished.

Wightman's formulation of QFT ⁹, although less economical from a conceptual point of view, is closer to the perturbative-theoretic framework and hence can also be used to investigate the mathematical structures at the basis of local gauge quantizations and to provide support to the perturbative methods. However, one has to face the problem that quantum gauge theories do not fit completely in such a formulation because of the presence of the Gauss law.

In fact, quantizations in “renormalizable” (local and covariant) gauges, based on unobservable fields, the Dirac field and the electromagnetic four-vector potential, necessarily yield an indefinite metric space \mathcal{G} of local states, containing vectors with no physical (quantum-mechanical) interpretation¹⁰. An additional constraint is then imposed in order to select physical states, following the Gupta-Bleuler (GB) formulation¹¹, and one has to face the problem that states with non-zero electric charge cannot exist in \mathcal{G} .

Quantization of electrodynamics in a gauge only involving physical degrees of freedom, as for instance the Coulomb gauge, imply on the other hand that the gauge fields cannot be neither local nor covariant and it is therefore difficult to set up a perturbative expansion and the renormalization procedure.

In Reference 12 it was shown that a possible strategy to determine physical charged states in the GB formulation is to obtain them as weak limits of local states, with the help of auxiliary topologies, introduced on \mathcal{G} in order to obtain weakly complete spaces of local states.

A different approach, pioneered by Dirac¹³ and subsequently developed by Symanzik¹⁴, is based on the (perturbative) construction of Coulomb-type fields, which yield physical states when applied to the ground state.

The main problem of this strategy is that in order to give a meaning to the formal exponent, a control of both ultraviolet and infrared problems is required. Such issues have been extensively discussed by Steinmann¹⁵, who proved the existence of physical charged fields for a large class of gauge-fixing functions within the framework of perturbation theory.

The characterization of the properties of charged states was discussed in Ref. 16, partially relying on Steinmann’s results; in this respect, it was proven that quantum effects imply non locality properties not expected from classical considerations, at least for the states obtained by applying suitably regularized exponentials to the vacuum.

The use of specific physical fields yields electrically charged states whose connection with the local and covariant formulation of QED is rather indirect. As a matter of fact, it demands to employ a non-trivial generalization of Feynman’s rules in the calculation of transition amplitudes and thus to take into account a number of additional contributions, with respect to the Feynman-Dyson perturbative theory, whose relevance for the formulation of scattering and for the outcome of practical computations is at present unclear. An example showing the non-triviality of this problem is Steinmann’s evaluation of the second-order radiative corrections to the magnetic moment of the electron; the proof that the result agrees with the value from standard perturbation theory involves

cancellations of a vast number of contributions¹⁷.

The results of the most recent investigations thus motivate on the one hand the need for a better understanding of the local and covariant formulation of QED ; on the other hand, they lead to a problem of minimality both in the classification of the charged states and in the use of physical charged fields.

In this paper we consider a hamiltonian (infrared) model in the Feynman-Gupta-Bleuler (FGB) gauge, based on an expansion of four-momenta whose relevance for the analysis of the infrared problem was first pointed out by Bloch and Nordsieck¹⁸.

In a previous work¹⁹, this model, which will be henceforth referred to as BN model, has been shown to allow to fully retrieve the results of the diagrammatic treatment of the infrared contributions of QED , in terms of the expansion of Möller operators obtained with the aid of an infrared cutoff, a mass renormalization and an adiabatic switching of the interaction.

The present paper is devoted to the study of the space-time properties of the BN model which can be inferred from the solution of the Heisenberg equations and to the construction and the characterization of (a class of) physical charged states²⁰.

First, we show that the GNS theorem over suitable product functionals ω_G , corresponding to vectors of the indefinite-metric Gupta-Bleuler space of the model, yields positive subspaces, a fact that will be important for the subsequent identification of physical charged states. Secondly, we establish the existence of Haag-Ruelle asymptotic limits for the four-vector potential. We find then that the Fock property, established²¹ for the representations of asymptotic algebras corresponding to massless bosons and associated to certain regions of Minkowsky space, in models of field theories in which locality and positivity are satisfied, is also enjoyed by the representations of the photon asymptotic algebras associated to the same regions and induced by ω_G , which fulfill locality but *not* positivity.

Concerning the construction of electrically charged states, two possible procedures will be outlined, both based on the realization of the Gauss law constraint via an automorphism of the observable algebra. The first procedure closely follows the strategy employed in classical electrodynamics and is based upon the existence of a solution of the free wave equation with support causally disjoint from that of the charge density; the second one is based on the introduction of a (suitably regularized) Dirac-type factor, constructed with the aid of the asymptotic electromagnetic fields.

As we shall see, since the requirement of a non-zero electric charge does not fix the automor-

phism uniquely, we obtain physical charged states with different large-distance behaviour, indexed by a superselected parameter related to the corresponding Liénard-Wiechert (LW) potentials.

Moreover, we show that such states define a charge class (a concept introduced by Buchholz in Ref. 3 within the algebraic setting), uniquely associated to the GB formulation. We also determine the properties of the representations of the asymptotic e.m. field algebras associated to suitably chosen lightcones and prove that the states of the GB charge class fulfill a notion of minimality, in the sense that they “only contain the photons associated to the asymptotic momentum of the particle”.

The plan of the manuscript is as follows.

In the first Section we establish notations, introduce the model and evaluate the solutions of the Heisenberg equations for the four-vector potential and the corresponding asymptotic e.m. fields.

In Section 2 we examine the localization properties of the expectations of product functionals over the four-vector potential and the asymptotic e.m. fields.

Section 3 is devoted to the construction and to the analysis of the properties of physical charged states with Liénard-Wiechert space-like asymptotics. First, we outline the Gupta-Bleuler strategy for the determination of solutions of Maxwell’s equations in classical electrodynamics and discuss a semiclassical argument concerning the classification of charged states. Afterwards, physical states are identified and their space-time features are investigated. We conclude the Section with a discussion regarding questions left open by our treatment, concerning the description of particles carrying an electric charge at asymptotic times and the vacuum-polarization effects.

1. NOTATIONS AND MAIN FEATURES OF THE MODEL

In this paper we will make use of the following notations.

The metric $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$ of Minkowski space is adopted and natural units are used ($\hbar = c = 1$). A four-vector is indicated with v^μ or simply with v , while the symbol \mathbf{v} denotes a three-vector; when confusion may arise, the notation \underline{v} will be employed. We use the symbol $c \cdot d$ for the indefinite inner product between four-vectors c and d .

The symbol A^\dagger stands for the hermitian conjugate of an operator A , defined on an indefinite-metric space, with respect to the indefinite inner product $\langle \cdot, \cdot \rangle$.

The commutator between two operators will be indicated by $[\cdot, \cdot]$. We denote by $a^\mu(\mathbf{k})$ and

$a^{\mu\dagger}(\mathbf{k})$ respectively the annihilation and creation operator-valued distributions in the *FGB* gauge, fulfilling the *CCR*

$$[a^{\mu}(\mathbf{k}), a^{\nu\dagger}(\mathbf{k}')] = -g^{\mu\nu} \delta(\mathbf{k} - \mathbf{k}').$$

In the same gauge, the Hamiltonian of the free e.m. field is

$$H_0^{e.m.} = - \int d^3k |\mathbf{k}| a^{\mu\dagger}(\mathbf{k}) a_{\mu}(\mathbf{k})$$

and the four-vector potential and the e.m. field tensor at time $t = 0$ are respectively

$$A^{\mu}(\mathbf{x}) \equiv \int \frac{d^3k}{\sqrt{2|\mathbf{k}|}} [a^{\mu}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} + a^{\mu\dagger}(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}}],$$

$$F^{\mu\nu}(\mathbf{x}) \equiv \partial^{\mu} A^{\nu}(\mathbf{x}) - \partial^{\nu} A^{\mu}(\mathbf{x}).$$

The symbols Ψ_F, ω_F, π_F will be used respectively for the no-particle vector of \mathcal{F} , the Fock vacuum functional and the Fock representation. The convolution with a form factor ρ is indicated by

$$A^{\mu}(\rho, \mathbf{x}) \equiv \int d^3\xi \rho(\xi) A^{\mu}(\mathbf{x} - \xi)$$

and for brevity we write

$$a(f(t)) \equiv \int d^3k a^{\mu}(\mathbf{k}) f_{\mu}(\mathbf{k}, t).$$

$\mathcal{S}(\mathbb{R}^3)$ will stand for the Schwartz space of C^{∞} functions of rapid decrease on \mathbb{R}^3 , $\mathcal{D}(\mathbb{R}^3)$ for the space of C^{∞} functions of compact support. Furthermore, we denote by \mathcal{A}_{obs} the observable algebra of the model and use the symbol $\mathcal{A}^{e.m.}$ for the subalgebra generated by $F^{\mu\nu}(\mathbf{x}, t)$.

The support of the convolution of ρ with a given charged-particle wave function is denoted by \mathcal{O} and its causal complement by \mathcal{O}' . The symbol \mathcal{O}_+ will stand for the region of Minkowski space formed by the points which have positive time-like distance from all the points of \mathcal{O} ; such a region will be referred to as the future tangent of \mathcal{O} , the past tangent \mathcal{O}_- being defined likewise, with obvious replacements. The algebra of observables associated to a given region \mathcal{C} of Minkowski space is denoted by $\mathcal{A}(\mathcal{C})$.

The system that we shall consider consists of a single charged non-relativistic quantum particle coupled to the quantum electromagnetic field and its dynamics is governed by the Hamiltonian

$$H^{(v)} \equiv \hat{\mathbf{p}} \cdot \mathbf{v} + H_0^{e.m.} + e v \cdot A(\rho, \hat{\mathbf{x}}) \equiv H_0^{(v)} + H_{int, \lambda}^{(v), (\hat{\mathbf{x}})}, \quad (1)$$

with $v \equiv (1, \mathbf{v})$ and $\rho \in \mathcal{S}(\mathbb{R}^3)$ a rotationally invariant distribution of charge, serving as ultraviolet cutoff.

The Heisenberg equations governing the dynamics of the algebra of the four-vector potential of the BN model are

$$\square A^\mu(\mathbf{x}, t) = j_{\hat{\mathbf{y}}}^\mu(\mathbf{x}, t), \quad (2)$$

where j^μ is the conserved charge-current density given by

$$j_{\hat{\mathbf{y}}}^\mu(\mathbf{x}, t) = e \left(\theta(-t) v^\mu \rho(|\mathbf{x} - \hat{\mathbf{y}} - \mathbf{v} t|) + \theta(t) v'^\mu \rho(|\mathbf{x} - \hat{\mathbf{y}} - \mathbf{v}' t|) \right). \quad (3)$$

\mathbf{v} is a triple of self-adjoint operators in a Hilbert space, to be identified as the observable corresponding to the asymptotic velocity of the particle. They commute with the Weyl algebra \mathcal{A}_{ch} generated by the canonical variables of the electron and with the polynomial algebras generated, in the Coulomb and Feynman gauge respectively, by the photon canonical variables. In the following, it will not be necessary to specify the detailed form of the interaction which changes the value of the \mathbf{v} -operators.

By taking the Fourier-transform, we can write down and easily solve (2), (3) in energy-momentum space. The equation of motion for the annihilation operator-valued distribution is

$$i \frac{d a^\mu(\mathbf{k}, t)}{d t} = |\mathbf{k}| a^\mu(\mathbf{k}, t) - \frac{\tilde{j}_{\hat{\mathbf{y}}}^\mu(\mathbf{k}, t)}{\sqrt{2} |\mathbf{k}|}, \quad (4)$$

with

$$\tilde{j}_{\hat{\mathbf{y}}}^\mu(\mathbf{k}, t) = e \tilde{\rho}(\mathbf{k}) \left(\theta(t) v'^\mu e^{-i \mathbf{k} \cdot (\hat{\mathbf{y}} + \mathbf{v}' t)} + \theta(-t) v^\mu e^{-i \mathbf{k} \cdot (\hat{\mathbf{y}} + \mathbf{v} t)} \right).$$

Since equation (4) is linear and non homogeneous, its solution can be written as the sum of the general solution of the free equation and of a solution of the non-homogeneous equation. For positive times,

$$\begin{aligned} a^\mu(\mathbf{k}, t) &= e^{-i |\mathbf{k}| t} \left[a^\mu(\mathbf{k}, t_0 = 0) + \frac{e v'^\mu \tilde{\rho}(\mathbf{k})}{\sqrt{2} |\mathbf{k}|} \frac{e^{i v' \cdot \mathbf{k} t} - 1}{v' \cdot \mathbf{k}} e^{-i \mathbf{k} \cdot \hat{\mathbf{y}}} \right] \\ &\equiv a_0^\mu(\mathbf{k}, t) + f_{v'}^\mu(\mathbf{k}, t; \hat{\mathbf{y}}). \end{aligned} \quad (5)$$

In order to obtain the solution for $t < 0$, it suffices to replace the final value of the four-velocity operator by the initial one. In the sequel, for definiteness we shall state the results for positive times. The space-time dependence of the four-vector potential can be computed by means of Fourier transformation, employing (5); one has

$$A^\mu(\mathbf{x}, t) = A_0^\mu(\mathbf{x}, t) + e F_{v'}^\mu(\mathbf{x}, t; \hat{\mathbf{y}}), \quad (6)$$

where A_0^μ satisfies $\square A_0^\mu(\mathbf{x}, t) = 0$ and the CCR

$$[A_0^\mu(\mathbf{x}, t), \dot{A}_0^\nu(\mathbf{x}', t)] = -i g^{\mu\nu} \delta(\mathbf{x} - \mathbf{x}'), \quad (7)$$

and F^μ is the (operator-valued) function

$$F_{v'}^\mu(\mathbf{x}, t; \hat{\mathbf{y}}) = \frac{1}{(2\pi)^{3/2}} \int d^3k e^{i\mathbf{k} \cdot \mathbf{x}} \tilde{F}_{v'}^\mu(\mathbf{k}, t; \hat{\mathbf{y}}), \quad (8)$$

$$\tilde{F}_{v'}^\mu(\mathbf{k}, t; \hat{\mathbf{y}}) \equiv \frac{1}{\sqrt{2|\mathbf{k}|}} [f_{v'}^\mu(\mathbf{k}, t; \hat{\mathbf{y}}) + \overline{f}_{v'}^\mu(-\mathbf{k}, t; \hat{\mathbf{y}})],$$

where $f_{v'}^\mu$ has been defined in (5).

The expression of $F_{v'}^\mu$ for $\mathbf{v}' = 0$ is denoted by $F_{v_0}^\mu$ and is calculated below. One has

$$F_{v_0}^\mu(\mathbf{x}, t; \hat{\mathbf{y}}) = \frac{1}{(2\pi)^{3/2}} \int d^3k e^{i\mathbf{k} \cdot (\mathbf{x} - \hat{\mathbf{y}})} \tilde{\rho}(\mathbf{k}) \frac{1 - \cos|\mathbf{k}|t}{|\mathbf{k}|^2} \delta^{\mu 0}. \quad (9)$$

In order to determine the localization of the support of (9), we note that the definition of the characteristic function χ_I of the interval I of the real line,

$$\chi_{[-x, x]}(y) = \frac{1}{\pi} \int_{-\infty}^{+\infty} d\xi e^{-i\xi y} \frac{\sin \xi x}{\xi}, \quad x > 0, \quad (10)$$

implies the equality

$$\frac{1}{(2\pi)^{3/2}} \int d^3k e^{i\mathbf{k} \cdot \mathbf{x}} \frac{1 - \cos|\mathbf{k}|t}{|\mathbf{k}|^2} = \sqrt{\frac{\pi}{2}} \frac{\chi_{|\mathbf{x}| < |t|}}{|\mathbf{x}|}. \quad (11)$$

Therefore (9) can be cast in the form

$$F_{v_0}^\mu(\mathbf{x}, t; \hat{\mathbf{y}}) = \int d^3z \rho(|\mathbf{x} - \hat{\mathbf{y}} - \mathbf{z}|) \chi_{|\mathbf{z}| < |t|} \frac{\delta^{\mu 0}}{4\pi|\mathbf{z}|}. \quad (12)$$

The result for a non-vanishing value \mathbf{v}' of the \mathbf{v} -operator could be obtained at once by the Lorentz covariance of the Gupta-Bleuler formulation, were it not for the fact that a non-covariant ultraviolet cutoff has been employed in the model. By introducing a Dirac delta, one can write the right-hand side (r.h.s.) of (12) as an integral over a volume element in Minkowski space; a change of integration variables in the resulting expression, performed by means of a boost corresponding to the four-velocity v' , then yields

$$F_{v'}^\mu(\mathbf{x}, t; \hat{\mathbf{y}}) = \int d^4z \rho(|\mathbf{x} - \hat{\mathbf{y}} - \underline{\Lambda}_{v'} z|) \delta((\Lambda_{v'} z)_0) \times \frac{v'^\mu \chi_{z^2 > 0}}{4\pi[(v' \cdot z)^2 - v'^2 z^2]^{1/2}}. \quad (13)$$

Apart from the convolution with the ρ function and from the presence of the Dirac delta, which accounts for the non covariance of the high-energy cutoff, the above expression equals the Liénard-Wiechert potential generated by a charge moving with a constant velocity \mathbf{v}' ; for negative times, the retarded potential is replaced by the advanced one.

We can now define and evaluate the asymptotic e.m. fields. Given a solution g of the free wave equation, such that $g(\mathbf{x}, t) \in \mathcal{S}(\mathbb{R}^3) \forall t$, we define

$$A^\mu(g)(x_0, t) \equiv \int d^3x \bar{g}(\mathbf{x}, x_0) \overleftrightarrow{\partial}_{x_0} A^\mu(\mathbf{x}, x_0 + t) \quad (14)$$

and the corresponding smearing of the (operator-valued) function, which appears in the solution (6),

$$F_v^\mu(g)(x_0, t; \hat{\mathbf{y}}) \equiv \int d^3x \bar{g}(\mathbf{x}, x_0) \overleftrightarrow{\partial}_{x_0} F_v^\mu(\mathbf{x}, x_0 + t; \hat{\mathbf{y}}), \quad (15)$$

where $f \overleftrightarrow{\partial}_{x_0} g \equiv f \partial_{x_0} g - (\partial_{x_0} f) g$. We give the calculation for $\mathbf{v} = 0$ and for the *out*-field, and write for brevity $F_{v_0} \equiv F_{v_0}^{\mu=0}$. Equations (12), (15), the properties of the test function and the Riemann-Lebesgue lemma lead to the existence of the limit (in the strong topology of multiplication operators)

$$\begin{aligned} F_{v_0}^{out}(g)(t; \hat{\mathbf{y}}) &\equiv \lim_{x_0 \rightarrow +\infty} F_{v_0}(g)(x_0, t; \hat{\mathbf{y}}) = - \int d^3k e^{-i \mathbf{k} \cdot \hat{\mathbf{y}}} \\ &\times \tilde{\rho}(\mathbf{k}) \left[\bar{g}(\mathbf{k}, x_0) \overleftrightarrow{\partial}_{x_0} \frac{\cos |\mathbf{k}| (x_0 + t)}{|\mathbf{k}|^2} \right] \Big|_{x_0=0} \\ &\equiv \int d^3x \left[\bar{g}(\mathbf{x}, x_0) \overleftrightarrow{\partial}_{x_0} G_{v_0}(\mathbf{x}, x_0 + t; \hat{\mathbf{y}}) \right] \Big|_{x_0=0}, \end{aligned} \quad (16)$$

with

$$G_{v_0}(\mathbf{x}, t; \hat{\mathbf{y}}) = - \int d^3z \rho(|\mathbf{x} - \hat{\mathbf{y}} - \mathbf{z}|) \frac{\chi_{|\mathbf{z}| > |t|}}{4\pi |\mathbf{z}|}. \quad (17)$$

It thus follows the existence of the *out*-field

$$\begin{aligned} A_{out}^\mu(g)(t) &\equiv \lim_{x_0 \rightarrow +\infty} A^\mu(g)(x_0, t) \equiv \int d^3x \left[\bar{g}(\mathbf{x}, x_0) \right. \\ &\times \overleftrightarrow{\partial}_{x_0} A_{out}^\mu(\mathbf{x}, x_0 + t) \Big] \Big|_{x_0=0}, \end{aligned} \quad (18)$$

$$A_{out}^\mu(\mathbf{x}, t) = A_{free}^\mu(\mathbf{x}, t) + G_{v_0}^\mu(\mathbf{x}, t; \hat{\mathbf{y}}), \quad (19)$$

with $G_{v_0}^\mu \equiv G_{v_0} \delta^{\mu 0}$.

We shall denote by \mathcal{F}^{as} the algebra of the electromagnetic observables constructed in terms of the asymptotic vector-potential A_{as} , $as = in, out$.

2. SPACE-TIME PROPERTIES OF THE LOCAL FORMULATION

The aim of this Section is to give an algebraic formulation to the relativistically covariant dynamics given by eqs.(2), (3) and to analyze the space-time properties of the expectations of product functionals, corresponding to vectors of the Gupta-Bleuler space of the model.

We recall that in the *FGB* gauge an observable is defined as a local function of the gauge fields which is left pointwise invariant by the residual symmetry group of the theory, the so-called gauge transformations of the second kind¹⁴. Equivalently, it can be identified by the condition that it commutes with the generator of such transformations, the $\partial \cdot A$ field, which is an observable in the above sense. In the sequel, it will be also denoted by B .

In the subsequent analysis it will be useful to consider the free electromagnetic algebras \mathcal{F}_0 and $\mathcal{A}_0^{e.m.}$ generated respectively by $A_0^\nu(\mathbf{x}, t)$ and $F_0^{\mu\nu}(\mathbf{x}, t) \equiv (\partial^\mu A_0^\nu - \partial^\nu A_0^\mu)(\mathbf{x}, t)$, which is left invariant by the gauge transformations of the second kind of the non-interacting theory and thus commutes with their generator $B_0 \equiv \partial \cdot A_0$.

The elements of $\mathcal{A}_0^{e.m.}$ thus fulfill

$$\begin{aligned} \partial_\mu F_0^{\mu\nu}(\mathbf{x}, t) &= -\partial^\nu B_0(\mathbf{x}, t), \quad \square B_0(\mathbf{x}, t) = 0, \\ \epsilon_{\mu\nu\rho\sigma} \partial^\nu F_0^{\rho\sigma}(\mathbf{x}, t) &= 0, \quad [F_0^{\rho\sigma}(x), B_0(x')] = 0, \end{aligned} \quad (20)$$

and the equal time canonical commutation relations induced by (7). Moreover, owing to (6), the generators of the restricted gauge transformations for $\mathcal{A}^{e.m.}$ and $\mathcal{A}_0^{e.m.}$ are related by

$$B(\mathbf{x}, t) = B_0(\mathbf{x}, t) + e(\partial \cdot F)(\mathbf{x}, t; \hat{\mathbf{y}}). \quad (21)$$

For $t > 0$ we also get the relation

$$F^{\mu\nu}(\mathbf{x}, t) = F_0^{\mu\nu}(\mathbf{x}, t) + H_v^{\mu\nu}(\mathbf{x}, t; \hat{\mathbf{y}}), \quad (22)$$

$$H_v^{\mu\nu}(\mathbf{x}, t; \hat{\mathbf{y}}) \equiv (\partial^\mu F_v^\nu - \partial^\nu F_v^\mu)(\mathbf{x}, t; \hat{\mathbf{y}}).$$

It is useful to recall that within the perturbative-theoretic treatment of *QED* in a local and covariant gauge the vacuum representation is required to be positive on the observables, as in a quantum field theory with positive-definite Wightman correlation functions. The outcome is an expansion around a non-interacting theory (in terms of renormalized parameters) and the vacuum representation is characterized by the choice of the Fock representation for the free four-vector potential.

Accordingly, in the model we shall assume a Fock representation for A_0^μ . For a given single particle state ω_{ch} , we consider the product functional

$$\omega_G \equiv \omega_{ch} \otimes \omega_F, \quad (23)$$

acting on the algebras \mathcal{A}_{ch} and \mathcal{F}_0 .

The space obtained via the *GNS* construction on ω_G is

$$\mathcal{G} \equiv L^2(\mathbf{y}; v) \otimes \mathcal{D}, \quad (24)$$

with L^2 the one-particle Hilbert space and \mathcal{D} the indefinite-metric Fock space obtained by applying polynomials of the free four-vector potential to the Fock vacuum.

Positivity of ω_F on $\mathcal{A}_0^{e.m.}$ implies that any vector belonging to the subspace of \mathcal{G} given by

$$\mathcal{K} \equiv L^2(\mathbf{y}; v) \otimes \mathcal{A}_0^{e.m.} \Psi_F \quad (25)$$

also defines a positive functional on $\mathcal{A}_0^{e.m.}$.

Moreover, \mathcal{K} is a space of physical states for $\mathcal{A}_0^{e.m.}$, since

$$\partial^\mu F_{0, \mu\nu}^{(-)}(\mathbf{x}, t) \Psi_F = 0 \quad (26)$$

implies

$$\partial^\mu F_{0, \mu\nu}^{(-)}(\mathbf{x}, t) \phi = 0, \quad \forall \phi \in \mathcal{K}, \quad (27)$$

and by the positivity of \mathcal{K} one also has

$$\partial^\mu F_{0, \mu\nu}(\mathbf{x}, t) = 0 \quad (28)$$

in \mathcal{K} .

It is important to stress that the functionals ω_G are positive on \mathcal{A}_{obs} , due to the fact that they are product functionals and to the explicit expression of the electromagnetic observables, given by (22). Of course, \mathcal{K} is not a space of physical states for \mathcal{A} , because the time evolution of the observable algebra does not obey the Maxwell equations; the deviation from Gauss' law is in fact given by

$$\omega_G(\partial_\mu F^{\mu\nu}(x)) = \omega_{ch}(j_{\hat{\mathbf{y}}}^\nu(x) - e \partial^\nu(\partial \cdot F)(x; \hat{\mathbf{y}})). \quad (29)$$

The previous relation holds for an arbitrary charged particle state, due to the fact that the negative-frequency component of the representative of B acts as a multiplication on the L^2 space of states

of the charge; as a matter of fact, it follows from (25), (21) that $\forall \psi \in L^2$

$$B^{(-)}(\mathbf{x}, t)(\psi(\mathbf{y}) \otimes \mathcal{A}_0^{e.m.} \Psi_F) = e(\partial \cdot F)^{(-)}(\mathbf{x}, t; \mathbf{y})(\psi(\mathbf{y}) \otimes \mathcal{A}_0^{e.m.} \Psi_F). \quad (30)$$

The space-time localization of the support of the eigenvalue on the r.h.s. of eq.(30) will be relevant in the characterization of the physical charged states carried out in the next Section; hence it is convenient to evaluate its explicit expression. It follows from (9) that

$$(\partial \cdot F)^{(-)}(\mathbf{x}, t; \mathbf{y}) = \int d^3z \rho(|\mathbf{x} - \mathbf{y} - \mathbf{z}|) \Delta^{(-)}(\mathbf{z}, t), \quad (31)$$

where $\Delta^{(-)}(x)$ is related to the Pauli-Jordan distribution

$$\Delta(\mathbf{x}, t) = \frac{1}{2\pi} \epsilon(t) \delta(t^2 - \mathbf{x}^2) \quad (32)$$

by $\Delta(x) = \Delta^{(-)}(x) - \Delta^{(-)}(-x)$.

Equations (29), (31) imply that Gauss' law is fulfilled by the restriction of the functionals (23) to the observables associated to \mathcal{O}_+ ; it follows that ω_G acts as a physical state on $\mathcal{A}(\mathcal{O}_+)$, since it is positive on \mathcal{A}_{obs} .

We remark that the validity of the weak Gauss law for the restriction of Gupta-Bleuler functionals to a forward lightcone is a non-perturbative feature of QED in the FGB gauge²²; the positivity of such a restriction remains instead an independent issue.

Concerning the space-like asymptotics in the charged (single-particle) sectors of the model, eqs.(6) and (13) imply, for a functional of the form (23), $\omega_G(A^\mu(\mathcal{O}')) = 0$. By the space-time localization properties of the electromagnetic observables, given by eqs.(22), (13), we get

$$\omega_G(\mathcal{A}^{e.m.}(\mathcal{O}')) = \omega_F(\mathcal{A}^{e.m.}(\mathcal{O}')). \quad (33)$$

We now turn to evaluate the expectations of the asymptotic radiation fields. One has

$$\omega_G(A_{out}^\mu(\mathbf{x}, t)) = \omega_G(G_{v_0}^\mu(\mathbf{x}, t; \hat{\mathbf{y}})). \quad (34)$$

For $\mathbf{v} = 0$, a Gupta-Bleuler product state thus yields expectations of A_{out}^μ in \mathcal{O}' which are equal in modulus to the Coulomb potential and have the opposite sign. The $1/r$ behaviour of (17) at space-like infinity is related to the fact that states with non-zero charge belonging to \mathcal{G} induce a non-Fock representation of the asymptotic vector potential; therefore, one can infer the occurrence of infinite photons at asymptotic times even for charged non-physical states. This result

agrees with those obtained on the basis of general hypotheses on the structure of local formulations of *QED* in Ref. 5 as well as in Ref. 12.

Furthermore, $\omega_G (A_{out}^\mu(\mathbf{x}, t))$ vanishes in $\mathcal{O}_+ \cup \mathcal{O}_-$ since, as follows from equation (12), there exists a frame of reference in which the interacting field is static in such a region. Therefore, one has in particular the additional piece of information that the corresponding representation of the subalgebra of the asymptotic photon fields constructed with the aid of the outgoing four-vector potential in \mathcal{O}_+ is Fock:

$$\omega_G (\mathcal{F}^{out}(\mathcal{O}_+)) = \omega_F (\mathcal{F}^{out}(\mathcal{O}_+)). \quad (35)$$

The consequences of this result for the classification of physical charged states will be discussed in Section 3.

For a non-vanishing $\mathbf{v} \equiv \mathbf{v}_{out}$,

$$A_{out}^\mu(\mathbf{x}, t) = A_{free}^\mu(\mathbf{x}, t) + G_{v_{out}}^\mu(\mathbf{x}, t; \hat{\mathbf{y}}), \quad (36)$$

$$G_{v_{out}}^\mu(\mathbf{x}, t; \hat{\mathbf{y}}) = - \int d^4 z \, \rho(|\mathbf{x} - \hat{\mathbf{y}} - \frac{\Lambda_{v_{out}}}{v_{out}} z|) \, \delta((\Lambda_{v_{out}} z)_0) \times \frac{v_{out}^\mu \chi_{z^2 < 0}}{4 \pi [(v_{out} \cdot z)^2 - v_{out}^2 z^2]^{1/2}}. \quad (37)$$

Besides (35), the localization of the support of $\omega_G (A_{out}^\mu(\mathbf{x}, t))$ also implies the Fock property for the representation of $\mathcal{F}^{out}(\mathcal{O}_-)$, confirming results established by Buchholz²¹ for models of field theories with massless bosons and “standard” charges (not satisfying a Gauss-law type constraint). We recall in fact that for such models the representations of boson asymptotic algebras, associated to any region of Minkowski-space, either bounded or unbounded, admitting a non-trivial future tangent, have been proven to be Fock; the same property is also fulfilled in the *BN* model by representations of the photon asymptotic algebras associated to the same regions and induced by (product) functionals corresponding to vectors of \mathcal{G} , for which locality, but not positivity, holds. The Fock property of $\mathcal{F}^{out}(\mathcal{O}_+)$ is instead an independent result of the model. Analogous statements apply to \mathcal{F}^{in} , with obvious changes.

For completeness we outline an argument explaining why one expects Fock representations for the asymptotic field algebras relative to massless bosons and associated to appropriately chosen regions of Minkowski space. Let \mathcal{C} be a space-time region with a non-trivial future tangent, π an irreducible positive-energy representation of the observable algebra and \mathcal{H} the associated *GNS*

space. By the Reeh-Schlieder theorem the vacuum is cyclic for the algebra $\mathcal{A}(\mathcal{C}_+)$, hence the vectors $\Psi = F \Omega$, with $F \in \mathcal{A}(\mathcal{C}_+)$ and Ω the ground-state vector, form a dense set \mathcal{V} of the Hilbert space \mathcal{H} .

On \mathcal{V} one has $A \Psi = A F \Omega = F A \Omega$, with A belonging to the center of $\pi(\mathcal{F}^{out}(\mathcal{C}))$, since by Huyghens' principle and locality $\mathcal{F}^{out}(\mathcal{C})$ is contained in $\mathcal{A}'(\mathcal{C}_+)$. The value taken by the elements of the center of $\pi(\mathcal{F}^{out}(\mathcal{C}))$ in a factorial representation thus equal those of the vacuum representation of the same subalgebra and $\pi(\mathcal{F}^{out}(\mathcal{C}))$ is quasi-equivalent to the Fock representation.²³

3. CONSTRUCTION AND PROPERTIES OF LIÉNARD-WIECHERT PHYSICAL CHARGED STATES

As a guide to the construction and of the classification of the charged physical states in the four-vector BN model, first we discuss the same problems in classical electrodynamics.

The electromagnetic fields generated by a charge-current distribution obey Maxwell's equations

$$\partial_\mu F^{\mu\nu} = j^\nu, \quad \partial^\mu F_{\mu\nu}^* = 0, \quad (38)$$

which can be expressed in terms of the vector potential:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \quad (39)$$

$$\square A^\mu - \partial^\mu \partial_\nu A^\nu = j^\mu.$$

Because of their invariance under gauge transformations,

$$A^\mu(\mathbf{x}, t) \rightarrow A^\mu(\mathbf{x}, t) + \partial^\mu \phi(\mathbf{x}, t), \quad \forall \phi(\mathbf{x}, t), \quad (40)$$

equations (39) admit a vast number of solutions; all vector potentials that are related to a given solution by a transformation as in (40) satisfy the same equations and are equally suited to describe a given physical configuration. In particular, equations (39) cannot be formulated in terms of canonical variables, since a solution can be changed into a different one by means of a gauge transformation not affecting the Cauchy data.

The set of solutions can be restricted by means of an auxiliary condition, the procedure being known as the choice of a gauge; for such a restriction not to lead to a loss of physical generality,

the requirement on the constraint is made that a vector potential that fulfills it can be obtained from an arbitrary one, by means of a gauge transformation.

An important property is that the solutions cannot be local, because of Gauss' law; this can be easily seen for example in the Coulomb gauge. The starting point for the canonical quantization is the formulation of classical electrodynamics in a local and covariant gauge; a gauge of this kind is singled out by the addition of a term to the Maxwell lagrangian, which, by weakening the Gauss law, allows for the existence of four-vector potentials obeying covariant field equations. In particular, the Feynman gauge is characterized by the equations of motion

$$\square A^\mu(\mathbf{x}, t) = j^\mu(\mathbf{x}, t) \quad (41)$$

and by the subsidiary condition

$$\partial^\mu (\partial \cdot A)(\mathbf{x}, t) = 0. \quad (42)$$

The idea is that in principle it might be simpler to solve the hyperbolic dynamics given by (41) and to deal with non locality at a later stage, when imposing the Gauss constraint, rather than to have to face directly the dynamical problem (39), involving non-local fields. The relevant point to specify is how condition (42) has to be imposed in order to allow to employ local solutions in the construction of potentials obeying Maxwell's equations. A solution of equation (41) will be called physical if it also satisfies (42).

The strategy at the basis of the Gupta-Bleuler formulation consists in first evaluating local solutions of (41), without constraints, and then in acting on them with suitable transformations in order to obtain solutions of (39). Starting from a given A_μ^G solving (41), one can obtain physical four-vector potentials A_μ^M by means of the transformation

$$A_\mu^M(\mathbf{x}, t) = A_\mu^G(\mathbf{x}, t) + C_\mu(\mathbf{x}, t), \quad (43)$$

C_μ being any free field whose four-divergence satisfies

$$(\partial^\mu C_\mu)(\mathbf{x}, t) = -(\partial^\mu A_\mu^G)(\mathbf{x}, t). \quad (44)$$

In classical electrodynamics the behaviour at space-like infinity implied by Maxwell's equations can thus be restored with the help of a free field, hence without changing the charge-current distribution. As discussed before, the support of C cannot be localized in a bounded space-time region. The above formulae also show that an assigned configuration of A^M at space-like infinity, described by a field C obeying (44), is compatible with a class of solutions of the wave equation,

differing by a divergence-less free field; this is related to the fact that the *FGB* formulation is invariant with respect to a residual group of local gauge transformations.

A convenient procedure to determine a physical solution with given space-like asymptotics is to consider $A_G^\mu = A_{(j)}^\mu$ among the solutions of the wave equation. One can then obtain a vector potential A^M solving the Maxwell equations for given Cauchy data, by means of the transformation (43), with $C_\mu = C_\mu^{(j)}$ fulfilling

$$(\partial^\mu C_\mu^{(j)})(\mathbf{x}, t) = -(\partial^\mu A_\mu^{(j)})(\mathbf{x}, t). \quad (45)$$

The Cauchy data of A^M determine C and viceversa:

$$C_\mu^{(j)}(\mathbf{x}, t=0) = A_\mu^M(\mathbf{x}, t=0), \quad (46)$$

$$\dot{C}_\mu^{(j)}(\mathbf{x}, t=0) = \dot{A}_\mu^M(\mathbf{x}, t=0).$$

For instance, one gets the Coulomb solution

$$A_\mu^{(Coul.)}(\mathbf{x}, t) \equiv A_\mu^{(j)}(\mathbf{x}, t) + C_\mu^{(Coul.)}(\mathbf{x}, t), \quad (47)$$

with $C_\mu^{(Coul.)}$ obeying the free wave equation with initial data

$$C_0^{(Coul.)}(\mathbf{x}, t=0) \equiv \int d^3y \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|} j_0(\mathbf{y}, t=0), \quad (48)$$

$$\dot{C}_0^{(Coul.)}(\mathbf{x}, t=0) \equiv 0, \quad (49)$$

$$C_i^{(Coul.)}(\mathbf{x}, t=0) = 0 = \dot{C}_i^{(Coul.)}(\mathbf{x}, t=0). \quad (50)$$

As a matter of fact, equation (50) also implies

$$\dot{C}_0^{(Coul.)}(\mathbf{x}, t=0) = -(\partial^\mu A_\mu^{(j)})(\mathbf{x}, t=0) = 0, \quad (51)$$

while (48) follows by

$$(\partial_0 \partial \cdot A^{(j)})(\mathbf{x}, t=0) = \square A_0^{(j)}(\mathbf{x}, t=0) = j_0(\mathbf{x}, t=0).$$

We can now discuss the features of the solutions of the classical system governed by equations (41), (42), corresponding to the charge-current density

$$j_\nu^\mu(\mathbf{x}, t) = e v^\mu \delta(\mathbf{x} - \mathbf{v} t). \quad (52)$$

A state of the system is determined by the Cauchy data for the four-vector potential and by the value of the velocity of the charge. The states can be arranged into classes according to their space-like asymptotics; a state is assigned to the class $\mathcal{C}_{c_{LW}}$, labeled by a time-like four-vector c_{LW} , if its initial data (satisfying the Gauss law constraint) can be written as

$$A^\mu(\mathbf{x}, t=0) = A_{c_{LW}}^\mu(\mathbf{x}, t=0) + g^\mu(\mathbf{x}, t=0), \quad (53)$$

$$\dot{A}^\mu(\mathbf{x}, t=0) = \dot{A}_{c_{LW}}^\mu(\mathbf{x}, t=0) + \dot{g}^\mu(\mathbf{x}, t=0).$$

In (53) $A_{c_{LW}}^\mu(\mathbf{x}, t=0)$, $\dot{A}_{c_{LW}}^\mu(\mathbf{x}, t=0)$ are the Cauchy data for the Liénard-Wiechert solution of equations (41), (42), with a constant velocity $\mathbf{v} = \mathbf{c}_{LW}$ in (52), and the second term on the right-hand side is a contribution with space-like infinity behaviour given by

$$\partial^\mu g^\nu(\mathbf{x}, t=0) - \partial^\nu g^\mu(\mathbf{x}, t=0) \sim o(\mathbf{x}^{-2}). \quad (54)$$

Each four-vector potential belonging to $\mathcal{C}_{c_{LW}}$ and solving equations (41), (42) for a velocity \mathbf{v} ($\mathbf{v} \neq \mathbf{c}_{LW}$) can be written as

$$A^\mu(\mathbf{x}, t) = A_{c_{LW}, v}^\mu(\mathbf{x}, t) + A_v^\mu(\mathbf{x}, t), \quad (55)$$

with A_v^μ the LW solution of eqs.(41), (42) for a current given by (52) and $A_{c_{LW}, v}^\mu$ a free four-vector potential. The corresponding free electromagnetic field

$$F_{c_{LW}, v}^{\mu\nu}(\mathbf{x}, t) \equiv \partial^\mu A_{c_{LW}, v}^\nu(\mathbf{x}, t) - \partial^\nu A_{c_{LW}, v}^\mu(\mathbf{x}, t) \quad (56)$$

satisfies the Maxwell equations, depends both on v and on c_{LW} and for each t has a non-compact support in position space, extending to space-like infinity; precisely, it decays as \mathbf{x}^{-2} . This field is therefore present in all states belonging to the class $\mathcal{C}_{c_{LW}}$, except for $\mathbf{v} = \mathbf{c}_{LW}$. Its functional dependence stems from the fact that by definition it must restore the Cauchy data for a state in $\mathcal{C}_{c_{LW}}$, since the initial conditions relative to A_v^μ in (55) belong to a different class.

The classes introduced for the classical system correspond to superselection sectors in the quantum theory. In particular, all the states of the class \mathcal{C}_c share the same space-like asymptotics, governed by the Liénard-Wiechert parameter c , and the representations of the observable algebra defined by states belonging to different classes are inequivalent, since the electric flux at space-like infinity takes a different value in each class and by locality cannot be modified by the application of observables.

Furthermore, as we shall see better below, the functional dependence of the classical radiation field $F_{c_{LW}, v}^{\mu\nu}$ corresponds to a non-Fock representation for the asymptotic electromagnetic algebra of the quantum system, if \mathbf{v} is interpreted as the asymptotic velocity of the charge. These arguments have been developed in the lectures on the infrared problem of Ref. 24 and have also been taken as a guide for non-relativistic *QED* in Ref. 25.

We can now take in consideration the quantum case. In the *GB* quantization, physical states are singled out by the auxiliary condition

$$(\partial \cdot A)^{(-)} \psi = 0. \quad (57)$$

Since in the *FGB* gauge $\partial \cdot A$ is a free field, its decomposition in negative and positive frequencies components is well defined.

For charged states, the existence of solutions of equation (57) in the Gupta-Bleuler space \mathcal{G} of local states, constructed with the aid of the local gauge fields, is excluded by the Gauss law and their construction involves a non-local procedure, which is far from trivial, as noted by Zwanziger²⁶. Such a construction substantially requires to appropriately implement conditions (43), (44). Actually, since locality of the charged fields is incompatible with positivity, the metric of \mathcal{G} is indefinite; therefore, the construction of physical charged states cannot rely on a completion of \mathcal{G} based on a standard Hilbert closure.

The discussion of main features of the *GB* formulation in classical electrodynamics, presented and the analysis carried out in the previous Section suggest how to proceed to construct physical charged states in the *BN* model.

The functionals corresponding to vectors of the space (25), and in particular those of the form $\omega_G \equiv \omega_\psi \otimes \omega_F$, are positive on \mathcal{A}_{obs} . In order to construct physical charged states, we introduce the automorphism of \mathcal{A}_{obs} defined by

$$\tilde{\alpha}(\hat{\mathbf{x}}) = \hat{\mathbf{x}}, \quad \tilde{\alpha}(v) = v, \quad (58)$$

$$\tilde{\alpha}(F^{\mu\nu}(\mathbf{x}, t)) = F^{\mu\nu}(\mathbf{x}, t) + e G^{\mu\nu}(\mathbf{x}, t; \hat{\mathbf{y}}), \quad (59)$$

$$(\partial_\mu G^{\mu\nu})(\mathbf{x}, t; \hat{\mathbf{y}}) = \partial^\nu (\partial \cdot F)(\mathbf{x}, t; \hat{\mathbf{y}}). \quad (60)$$

The term on the r.h.s. of (60) involves the four-divergence of the (operator-valued) four-vector function F_v^μ , given by eq.(13), and $\hat{\mathbf{y}}$ is the position operator of the charged particle. By equations (59), (60) and by the definition of \mathcal{K} , we conclude that the functional $\tilde{\omega}_\phi$, with expectations

given by

$$\tilde{\omega}_\phi(\mathcal{A}_{obs}) \equiv \omega_\phi(\tilde{\alpha}(\mathcal{A}_{obs})) = \langle \phi, \tilde{\alpha}(\mathcal{A}_{obs}) \phi \rangle, \quad (61)$$

is a physical charged state, $\forall \phi \in \mathcal{K}$.

The classification of such states is most easily done by employing the formulation in terms of quantum vector potentials. The automorphism (59), (60) is induced by the transformation

$$\tilde{A}^\mu(\mathbf{x}, t) \equiv A^\mu(\mathbf{x}, t) + e C^\mu(\mathbf{x}, t; \hat{\mathbf{y}}), \quad (62)$$

with C^μ a multiplication operator in the particle space, obeying a free dynamics in the variables \mathbf{x}, t and with four-divergence fulfilling

$$(\partial \cdot C)(\mathbf{x}, t; \hat{\mathbf{y}}) = -(\partial \cdot F)(\mathbf{x}, t; \hat{\mathbf{y}}). \quad (63)$$

In analogy with the classical theory, the Coulomb solution can be obtained by solving (63) with the (operatorial) conditions $C_i(\mathbf{x}, t=0; \hat{\mathbf{y}}) = 0 = \dot{C}_i(\mathbf{x}, t=0; \hat{\mathbf{y}})$. By means of the transformation

$$\tilde{A}_{Coul.}^\mu(\mathbf{x}, t) \equiv A^\mu(\mathbf{x}, t) + C_{Coul.}^\mu(\mathbf{x}, t; \hat{\mathbf{y}}), \quad (64)$$

with (by also employing eq.(11))

$$C_{Coul.}^\mu(\mathbf{x}, t; \hat{\mathbf{y}}) = \int d^3z \rho(|\mathbf{x} - \hat{\mathbf{y}} - \mathbf{z}|) \frac{\chi_{|\mathbf{z}| > |t|}}{4\pi|\mathbf{z}|} \delta^{\mu 0}, \quad (65)$$

one defines

$$\omega_{Coul.}(A^\mu(\mathbf{x}, t)) \equiv \omega_G(\tilde{A}_{Coul.}^\mu(\mathbf{x}, t)). \quad (66)$$

By employing the covariance of (63), which holds apart from the ultraviolet cutoff, we can construct physical charged states with LW space-like asymptotics; with the aid of the transformation

$$\tilde{A}_{LW}^\mu(\mathbf{x}, t) \equiv A^\mu(\mathbf{x}, t) + C_{c_{LW}}^\mu(\mathbf{x}, t; \hat{\mathbf{y}}), \quad (67)$$

$$C_{c_{LW}}^\mu(\mathbf{x}, t; \hat{\mathbf{y}}) = \int d^4z \rho(|\mathbf{x} - \hat{\mathbf{y}} - \frac{\Lambda_{c_{LW}}}{\Lambda_{c_{LW}}} z|) \delta((\Lambda_{c_{LW}} z)_0) \times \frac{c_{LW}^\mu \chi_{z^2 < 0}}{4\pi[(c_{LW} \cdot z)^2 - c_{LW}^2 z^2]^{1/2}}, \quad (68)$$

one obtains physical charged states labeled by the time-like Liénard-Wiechert four-vector c_{LW} :

$$\omega_{LW}(A^\mu(\mathbf{x}, t)) \equiv \omega_G(\tilde{A}_{LW}^\mu(\mathbf{x}, t)). \quad (69)$$

Such states, which will be referred to in the sequel as Liénard-Wiechert states, are constructed via an automorphism which amounts to a shift of the contribution from the hyperbolic dynamics by a non-local solution of the free wave equation, in analogy with the classical theory. The field (68) obeys indeed a free evolution, because it is the convolution with (the Lorentz-transformed of) a (Coulomb) field obeying a free wave equation.

It is interesting to point out that, although they are not obtained from vectors of \mathcal{G} through a limiting procedure, the states (69) are nevertheless quite linked to the indefinite-metric space, since their expectations on the subalgebras $\mathcal{A}(\mathcal{O}_+)$ and $\mathcal{A}(\mathcal{O}_-)$ equal those of ω_G . Such a property of the LW states ultimately relies on the existence of solutions of the free wave equation with support in \mathcal{O}' .

Now we wish to analyze how states with LW asymptotics can be constructed by means of a procedure employing the asymptotic electromagnetic fields. Let us introduce the operators (for definiteness, we consider the *out* vector potential, the treatment for the *in* field being analogous, with obvious changes)

$$\begin{aligned} U_R(C) &\equiv \exp\left(-ie \int d^3x A_{out}^\mu(\mathbf{x}, t) \overleftrightarrow{\partial}_t C_{R,\mu}(\mathbf{x} - \hat{\mathbf{y}}, t)\right) = \\ &= \lim_{t \rightarrow +\infty} \exp\left(-ie \int d^3x A^\mu(\mathbf{x}, t) \overleftrightarrow{\partial}_t C_{R,\mu}(\mathbf{x} - \hat{\mathbf{y}}, t)\right), \end{aligned} \quad (70)$$

where C_R is obtained by regularizing the field C , introduced in (62), (63) and acting as a multiplication operator in the charged particle space, as follows

$$C_R^\mu(\mathbf{x} - \hat{\mathbf{y}}, t) \equiv C^\mu(\mathbf{x} - \hat{\mathbf{y}}, t) \chi_R(\mathbf{x} - \hat{\mathbf{y}}), \quad (71)$$

$$\chi_R(\mathbf{x}) \equiv \chi\left(\frac{|\mathbf{x}|}{R}\right), \quad R > 0, \quad (72)$$

with χ a smooth function with compact support. The functions (72) serve to cope with the infrared divergences of the smeared field caused by the long-range Coulomb tail of the test functions, which must necessarily be present for the Gauss law to be fulfilled.

With the aid of those operators, one can introduce a one-parameter group of automorphisms of \mathcal{F}^{out} ,

$$\alpha_R(D) \equiv U_R^{-1}(C) D U_R(C), \quad D \in \mathcal{F}^{out}, \quad (73)$$

induced by the transformation

$$U_R^{-1}(C) A_{out}^\mu(\mathbf{x}, t) U_R(C) = A_{out}^\mu(\mathbf{x}, t) + e C_R^\mu(\mathbf{x} - \hat{\mathbf{y}}, t), \quad (74)$$

and by the corresponding one on the time-derivative of the four-vector potential. In particular, one has

$$\alpha_R((\partial \cdot A)(\mathbf{x}, t)) = (\partial \cdot A)(\mathbf{x}, t) + e(\partial \cdot C_R)(\mathbf{x}, t). \quad (75)$$

The functionals defined on \mathcal{F}^{out} as

$$\omega_{\psi, R}(D) \equiv \langle U_R \psi, D U_R \psi \rangle, \quad \psi \in \mathcal{K}, \quad D \in \mathcal{F}^{out}, \quad (76)$$

are positive, since

$$\begin{aligned} \omega_{\psi, R}(D^* D) &= \langle U_R \psi, D^* D U_R \psi \rangle = \langle \psi, \alpha_R(D^* D) \psi \rangle \\ &= \omega_{\psi}(\alpha_R(D^* D)) \geq 0. \end{aligned}$$

Concerning the removal of the infrared cutoff, one needs to specify a notion of convergence. Let $\alpha(D) \equiv \lim_{R \rightarrow +\infty} \alpha_R(D)$; the infrared cutoff can then be removed in the expectations of asymptotic observables, yielding

$$\omega_{\psi}(D) \equiv \lim_{R \rightarrow +\infty} \omega_{\psi, R}(D) = \omega_{\psi}(\alpha(D)). \quad (77)$$

The limiting functionals are positive,

$$\omega_{\psi}(D^* D) = \lim_{R \rightarrow +\infty} \omega_{\psi, R}(D^* D) \geq 0, \quad (78)$$

and are physical states, since, by (75), (63) and the definition of \mathcal{K} ,

$$\omega_{\psi}((\partial \cdot A)(\mathbf{x}, t)) = \lim_{R \rightarrow +\infty} \omega_{\psi}(\alpha_R((\partial \cdot A)(\mathbf{x}, t))) = 0. \quad (79)$$

A relevant property of Dirac-type factors as (70) is that by Huyghens' principle they preserve the expectations of e.m. observables within forward and backward lightcones, since they are obtained as (suitably regularized) time limits of the interacting four-vector potential, smeared with solutions of the free wave equation. It would be of interest to investigate whether the use of such exponentials, which can be regarded as alternative with respect to the approach based on the construction of Dirac-type exponentials by means of interacting gauge fields^{15,16}, may be successfully extended to *QED*.

In the sequel we show that the LW charged states, which belong to superselection sectors labeled by different values of c_{LW} , can be arranged in a charge class, a concept introduced by Buchholz³ in a general analysis of the state space of Quantum Electrodynamics.

We recall that positive energy representations of the observable algebra, with given electric charge, factorial in a forward lightcone V_+ and possibly belonging to different superselection sectors, are assigned to the same charge class if their restrictions to V_+ are equivalent.

This concept is physically motivated by the kinematic consideration that since electrically charged particles are massive they have to eventually enter any forward lightcone and thus it should be possible to determine the total charge of a state by measurements performed in such a region. In this way one should be able to distinguish the electric charge among the superselection rules in QED , since, due to the possible presence of photons coming from asymptotic negative times, measurements in a forward lightcone are not enough to determine the value of the electric flux at space-like infinity in a given representation; each charge class should then contain superselection sectors with a given electric charge but different flux-distributions.

Buchholz showed that given an irreducible positive energy representation π of the observable algebra in a Hilbert space \mathcal{H} , the restrictions to the subalgebras $\mathcal{F}^{out}(V_+)$ of the representations belonging to $[\pi]$ are equivalent; therefore, the sectors in a charge class cannot be distinguished by measurements of the outgoing electromagnetic fields in the forward lightcone.

He also proved that by adding an arbitrary number of low-energy photons to a given $\Psi \in \mathcal{H}$ one can construct a state, with finite energy and the same charge, inducing a representation of $\mathcal{F}^{out}(V_+)$ inequivalent to $\pi(\mathcal{F}^{out}(V_+))$ and thus corresponding to a different charge class. Therefore, charge classes are in general characterized not only by the total electric charge of their states, but also by the presence of background radiation fields. A criterion of infrared minimality can be introduced, by demanding that it selects the charge classes whose superselection sectors have the best possible localization properties with respect to the vacuum, that is, they do not have any background radiation field. Such sectors are expected to be a convenient set for a systematic analysis of the infrared problem.

We shall now discuss the space-time properties of the LW physical charged states of the model. First, such states coincide on $\mathcal{A}(\mathcal{O}_+)$, since the free field C^μ has support in \mathcal{O}' , and are given by ω_G , which is positive and satisfies the auxiliary condition in \mathcal{O}_+ ; product functionals defined within the GB formulation identify therefore a *unique* charge class. Secondly, equations (67), (68) imply that the LW states induce Fock representations of the subalgebras $\mathcal{F}^{out}(\mathcal{O}_+)$ and $\mathcal{F}^{out}(\mathcal{O}_-)$; this gives the simplest result for the indetermination left by Buchholz's treatment for the representations of $\mathcal{F}^{as}(\mathcal{O}_+)$ and $\mathcal{F}^{as}(\mathcal{O}_-)$.

The Gupta-Bleuler charge class can thus be identified as the charge class containing all repre-

representations π of the algebra of observables whose restrictions to $\mathcal{F}^{out}(\mathcal{O}_+)$ and $\mathcal{F}^{out}(\mathcal{O}_-)$ are Fock. Since this result can be seen as a consequence of the locality of the gauge fields and of the support properties of the automorphism on the asymptotic electromagnetic algebras, one expects that it might also hold in the corresponding formulation of *QED*.

It is worthwhile to remark that the condition on $\mathcal{F}^{out}(\mathcal{O}_+)$ agrees with the above-mentioned result that the representations of the outgoing electromagnetic field in a charge class cannot be distinguished by measurements in the forward lightcone. The Fock property of $\pi(\mathcal{F}^{out}(\mathcal{O}_+))$ also implies that the *GB* charge class is infrared minimal in the sense described above, namely, it does not contain background radiation fields but only the field associated to the asymptotic momentum of the charge.

We also point out that the condition on $\mathcal{F}^{out}(\mathcal{O}_-)$ is due to the fact that the *LW* states have been constructed with the aid of a free field with expectations having support in \mathcal{O}' , and therefore enjoy at least some of the locality properties holding in models of field theories with massless bosons and standard charges.

We conclude the paper with a discussion concerning the question of how the model should be improved in order to account for a proper description of electrically charged particle at large times and of fermion-loop effects.

Concerning the first issue, the algebraic theory provides the following general framework. The elements of the algebra \mathcal{M}^{out} of the observables describing the (charged) massive particles at positive asymptotic times (and therefore associated to the intersection of all forward lightcones) are compatible with the observables of \mathcal{F}^{out} , owing to Huyghens' principle. Such a property should hold in particular for the asymptotic four-velocity of the charge, which it is expected to belong to the center of \mathcal{F}^{out} , namely to index inequivalent representations of this algebra, as it happens in the *BN* model.

Under this assumption, the representation of the algebra of all outgoing observables \mathcal{A}^{out} can be reduced with respect to the asymptotic momenta of the charges and cannot therefore provide a complete characterization of the physical system. This is related to the fact that an observable describing the asymptotic position of a particle carrying an electric charge cannot exist, if it is requested to eventually enter any forward lightcone; in fact, by kinematical reasons such an observable would necessarily belong to $\mathcal{F}^{out'}$ and thereby commute with the corresponding asymptotic momentum.

Buchholz suggested that a collision theory employing a complete set of variables for the out-

going charged particles may be achieved if one considers observables contained in a *fixed* forward lightcone. These considerations agree with the conjecture that it may be possible to completely determine a scattering matrix by considering a modified asymptotic dynamics for the charges, allowing for the construction of an asymptotic position variable, along the lines of Dollard's treatment of Coulomb scattering²⁷. In fact, the dynamics of the position variable at large (positive) times has to account for the effects of the interaction with photons, which are not described by the observables belonging to the intersection of all (forward) lightcones.

Concerning the fermion-loop effects, a general theorem states¹⁶ that for any subalgebra C of local observables, stable under translations and irreducible in the vacuum sector, the quantum corrections cannot vanish for all elements of C ; in particular, it seems to indicate that, as consequence of the delocalization caused by vacuum polarization effects, there cannot exist electrically charged states which are local with respect to the charge-current density.

As discussed before, in the BN model the behaviour at space-like infinity implied by Maxwell's equations can be restored with the help of a free field, hence without changing the charge-current distribution; the so-obtained LW states are thus local with respect to the charge-current density. Of course, this property is not in contradiction with the above-mentioned theorem, since the model does not account for loops of fermions, but the point is that the same feature seems to be necessarily implied by locality also in QED , for physical charged states constructed within the GB formulation²⁸. However, since well-known results from the standard diagrammatic expansion imply that contributions from fermion loops should vanish for asymptotic times, the relevance of the theorem for electrically charged states constructed with the aid of asymptotic fields is unclear. This problem certainly deserves further study.

ACKNOWLEDGMENTS

This paper is based in part on preliminary investigations by G. Morchio and F. Strocchi on the space-time properties of solvable hamiltonian models, devoted to a better understanding of the Gupta-Bleuler formulation of Quantum Electrodynamics. I would like to thank G. Morchio for extensive discussions on these topics.

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- ²²This fact follows by the relations among gauge fields inferred by Symanzik¹⁴ from the canonical commutation relations and the equations of motion and has been pointed out to me by G. Morchio and F. Strocchi.
- ²³Actually, $\pi (\mathcal{F}^{out} (\mathcal{C}))$ can be shown²¹ to be unitarily equivalent to π_F as a consequence of the fact that the vacuum is a separating vector for $\mathcal{F}^{out} (\mathcal{C})$ and of a theorem on normal states from the theory of Von Neumann’s algebras.
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